



Chapter 4

Primal-Dual Algorithms

Convex Optimization for Computer Vision
SS 2016

Recap

PDHG

Algorithm

Primal-dual gap

Convergence

Applications

Modifications

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Recap

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Recalling gradient methods

Structured problems

1 Gradient descent:

$$\min_u E(u)$$

for $E : \mathbb{R}^n \rightarrow \mathbb{R}$ L-smooth: $\mathcal{O}(1/k)$,

2 Subgradient descent:

$$\min_u E(u)$$

$E : \mathbb{R}^n \rightarrow \mathbb{R}$ Lipschitz continuous, stepsizes $\rightarrow 0$: $\mathcal{O}(1/\sqrt{k})$.

3 Proximal gradient:

$$\min_u F(u) + G(u)$$

$F : \mathbb{R}^n \rightarrow \mathbb{R}$ L-smooth, $G : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$ simple: $\mathcal{O}(1/k)$.

3* Gradient projection: Special case of prox. grad. for $G = \iota_C$

Strong convexity: Linear convergence $\mathcal{O}(c^k)$, $c < 1$, of **1** and **3**.





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Structured problems

How would you solve

$$\min_u \frac{1}{2} \|u - f\|_1 + \alpha \|Ku\|_2^2$$

→ Proximal gradient

How would you solve

$$\min_u \frac{1}{2} \|u - f\|^2 + \alpha \|Ku\|_2^2$$

Gradient Descent (although there are better ways).

How would you solve

$$\min_u \frac{1}{2} \|u - f\|^2 + \alpha \|Ku\|_1$$

→ Derive dual problem, apply gradient projection

Structured problems

How would you solve

$$\min_u \frac{1}{2} \|u - f\|_1 + \alpha \|Ku\|_1$$

→ Subgradient descent

Is this really the best we can do for such a problem? No!

Very important class of algorithms we have not considered yet!

Applicable to

$$\min_u G(u) + F(Ku)$$

with $G : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$ simple¹, $F : \mathbb{R}^m \rightarrow \mathbb{R} \cup \{\infty\}$ simple,
 $K : \mathbb{R}^n \rightarrow \mathbb{R}^m$ linear.

¹simple = easy to evaluate proximity operator





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The convex-conjugate of E is

$$E^*(p) = \sup_u \langle u, p \rangle - E(u)$$

Fact 1: For a proper, closed, convex E it holds that

$$E = E^{**}$$

Trick to optimize the TV functional: Use

$$\|Ku\|_{2,1} = (\|\cdot\|_{2,1})^{**}(Ku) = \sup_{\|p\|_{2,\infty} \leq 1} \langle Ku, p \rangle$$

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Proximal operator

$$\text{prox}_E(v) := \underset{u}{\operatorname{argmin}} E(u) + \frac{1}{2} \|u - v\|^2$$

Moreau decomposition:

$$v = \text{prox}_E(v) + \text{prox}_{E^*}(v)$$

Fact 2: If E is simple, E^* is simple, too!

Fact 3: The operator $\text{prox}_{\tau E}$ has the interpretation of an implicit gradient descent step (even for nondifferentiable functions).

Opposed to explicit (sub-)gradient descent it guarantees the reduction of E for any τ



Primal-dual formulation

Let us consider

$$\min_u G(u) + F(Ku)$$

with G and F being simple.

Let us try what helped us in our first computation for deriving duality: Use "fact 1": $F = F^{**}$

$$\min_u G(u) + F(Ku) = \min_u \sup_p G(u) + \langle Ku, p \rangle - F^*(p)$$

Based on "fact 2", prox_{F^*} is easy to evaluate.

Let's try to alternate between an implicit gradient descent on u and an implicit gradient ascent in p !



Primal-dual formulation

Define

$$PD(u, p) := G(u) + \langle Ku, p \rangle - F^*(p)$$

and try

$$p^{k+1} = \text{prox}_{-\sigma PD(u^k, \cdot)}(p^k),$$

$$u^{k+1} = \text{prox}_{\tau PD(\cdot, p^{k+1})}(u^k),$$

One finds

$$\begin{aligned} p^{k+1} &= \text{prox}_{-\sigma PD(u^k, \cdot)}(p^k), \\ &= \underset{p}{\text{argmin}} \frac{1}{2} \|p - p^k\|^2 + \sigma F^*(p) - \sigma \langle Ku^k, p \rangle \\ &= \underset{p}{\text{argmin}} \frac{1}{2} \|p - p^k - \sigma Ku^k\|^2 + \sigma F^*(p) \\ &= \text{prox}_{\sigma F^*}(p^k + \sigma Ku^k) \end{aligned}$$



Primal-dual formulation

Define

$$PD(u, p) := G(u) + \langle Ku, p \rangle - F^*(p)$$

and try

$$p^{k+1} = \text{prox}_{\sigma F^*}(p^k + \sigma Ku^k),$$

$$u^{k+1} = \text{prox}_{\tau PD(\cdot, p^{k+1})}(u^k),$$

One finds

$$\begin{aligned} u^{k+1} &= \text{prox}_{\tau PD(\cdot, p^{k+1})}(u^k), \\ &= \underset{u}{\text{argmin}} \frac{1}{2} \|u - u^k\|^2 + G(u) + \langle Ku, p^{k+1} \rangle \\ &= \underset{u}{\text{argmin}} \frac{1}{2} \|u - u^k + \tau K^* p^{k+1}\|^2 + \tau G(u) \\ &= \text{prox}_{\tau G}(u^k - \tau K^* p^{k+1}) \end{aligned}$$



Primal-dual hybrid gradient method

We found

$$\begin{aligned}p^{k+1} &= \text{prox}_{\sigma F^*}(p^k + \sigma K u^k), \\u^{k+1} &= \text{prox}_{\tau G}(u^k - \tau K^* p^{k+1}).\end{aligned}$$

One should make one (currently unintuitive) modification:

$$\begin{aligned}p^{k+1} &= \text{prox}_{\sigma F^*}(p^k + \sigma K \bar{u}^k), \\u^{k+1} &= \text{prox}_{\tau G}(u^k - \tau K^* p^{k+1}), \\ \bar{u}^{k+1} &= 2u^{k+1} - u^k.\end{aligned} \quad (\text{PDHG})$$

We will understand this modification very well in about 2 weeks!

Our goal: Prove that the **Primal-Dual Hybrid Gradient Method**² (PDHG) converges!

²Pock, Cremers, Bischof, Chambolle '08, Esser, Zhang, Chan '09, Chambolle, Pock '10



Saddle points

We wrote

$$\min_u G(u) + F(Ku) = \min_u \sup_p PD(u, p)$$

where

$$PD(u, p) = G(u) + \langle Ku, p \rangle - F^*(p)$$

for proper, closed, convex G and F . We assume that a minimizer \tilde{u} exists. Is the "sup" attained for some \tilde{p} , too?

Definition

Let us call (\tilde{u}, \tilde{p}) a *saddle-point* of $PD : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \bar{\mathbb{R}}$, if

$$PD(\tilde{u}, p) \leq PD(\tilde{u}, \tilde{p}) \leq PD(u, \tilde{p})$$

holds for all $u \in \mathbb{R}^n, p \in \mathbb{R}^m$.



Existing saddle-point

1. Let there exist $\tilde{u} \in \operatorname{argmin}_u G(u) + F(Ku)$,
2. Let there exist a $u \in \operatorname{ri}(\operatorname{dom}(G))$ such that $Ku \in \operatorname{ri}(\operatorname{dom}(F))$.

Then (according to the sum rule)

$$0 \in \partial G(\tilde{u}) + K^* \partial F(K\tilde{u}).$$

In particular, F is subdifferentiable at $K\tilde{u}$ and we know that

$$\sup_p \langle K\tilde{u}, p \rangle - F^*(p) = F^{**}(K\tilde{u}) = \langle K\tilde{u}, \tilde{p} \rangle - F^*(\tilde{p})$$

for $\tilde{p} \in \partial F(K\tilde{u})$ according to the Fenchel-Young inequality.

→ Under 1. and 2., a saddle point of PD exists!





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Partial primal-dual gap

Definition

Given two compact sets B_1 and B_2 , we call

$$\mathcal{G}_{B_1 \times B_2}(u, p) = \max_{p' \in B_2} PD(u, p') - \min_{u' \in B_1} PD(u', p)$$

the *partial primal-dual gap*.

Properties of the partial primal-dual gap

Let (\tilde{u}, \tilde{p}) be a saddle point of the min-max problem

$$\min_u \max_p G(u) + \langle Ku, p \rangle - F^*(p),$$

and let $(\tilde{u}, \tilde{p}) \in B_1 \times B_2$. Then

$$\mathcal{G}_{B_1 \times B_2}(u, p) \geq 0 \quad \forall (u, p) \in \mathbb{R}^n \times \mathbb{R}^m$$

$$\mathcal{G}_{B_1 \times B_2}(u, p) = 0 \Leftrightarrow (u, p) \text{ is a saddle point.}$$

PDHG convergence theorem

Theorem (Chambolle-Pock '10)

In addition to the previous assumptions, let $L = \|K\|$, let $\tau\sigma L^2 < 1$, and let (p^n, u^n, \bar{u}^n) be the iterates of (PDHG) for an arbitrary starting point u^0, p^0 and $\bar{u}^0 = u^0$.

Then (u^n, p^n) converge to a saddle-point (u^*, p^*) of PD . For $u_N = (\sum_{k=1}^N u^k)/N$, $p_N = (\sum_{k=1}^N p^k)/N$, and any compact $B_1 \times B_2 \subset \mathbb{R}^n \times \mathbb{R}^m$, it holds that

$$\mathcal{G}_{B_1 \times B_2}(u_N, p_N) \leq \frac{D(B_1, B_2)}{N},$$

where

$$D(B_1, B_2) = \max_{(u,p) \in B_1 \times B_2} \frac{\|u - u^0\|^2}{2\tau} + \frac{\|p - p^0\|^2}{2\sigma},$$

and (u_N, p_N) also converge to (u^*, p^*) .

Proof: Board.





The primal-dual hybrid gradient method

$$p^{k+1} = \text{prox}_{\sigma F^*}(p^k + \sigma K \bar{u}^k),$$

$$u^{k+1} = \text{prox}_{\tau G}(u^k - \tau K^* p^{k+1}), \quad (\text{PDHG})$$

$$\bar{u}^{k+1} = u^{k+1} + (u^{k+1} - u^k).$$

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ROF Denoising

$$\min_u P(u) = \min_u \frac{1}{2} \|u - f\|^2 + \alpha \|Ku\|_{2,1}$$

with K being a discretization of the multichannel gradient operator.



ROF Denoising

We write

$$\min_u P(u) = \min_u \max_p \frac{1}{2} \|u - f\|^2 + \langle Ku, p \rangle - \iota_{\|\cdot\|_{2,\infty} \leq \alpha}(p).$$

The (PDHG) updates are

$$\begin{aligned} p^{k+1} &= \operatorname{prox}_{\sigma F^*}(p^k + \sigma K \bar{u}^k) \\ u^{k+1} &= \operatorname{prox}_{\tau G}(u^k - \tau K^* p^{k+1}), \\ \bar{u}^{k+1} &= 2u^{k+1} - u^k. \end{aligned}$$

which in this case amounts to

$$\begin{aligned} p^{k+1} &= \operatorname{argmin}_p \frac{1}{2} \|p - (p^k + \sigma K \bar{u}^k)\|^2 + \sigma \iota_{\|\cdot\|_{2,\infty} \leq \alpha}(p), \\ u^{k+1} &= \operatorname{argmin}_u \frac{1}{2} \|u - (u^k - \tau K^* p^{k+1})\|^2 + \frac{\tau}{2} \|u - f\|^2 \\ &= \frac{u^k - \tau K^* p^{k+1} + \tau f}{1 + \tau} \\ \bar{u}^{k+1} &= 2u^{k+1} - u^k. \end{aligned}$$



TV- L^1 Denoising

$$\min_u P(u) = \min_u \|u - f\|_1 + \alpha \|Ku\|_{2,1}$$

with K being a discretization of the multichannel gradient operator.





We write

$$\min_u P(u) = \min_u \max_p \frac{1}{2} \|u - f\|_1 + \langle Ku, p \rangle - \iota_{\|\cdot\|_2, \infty \leq \alpha}(p).$$

The (PDHG) updates are

$$\begin{aligned} p^{k+1} &= \text{prox}_{\sigma F^*}(p^k + \sigma K \bar{u}^k) \\ u^{k+1} &= \text{prox}_{\tau G}(u^k - \tau K^* p^{k+1}), \\ \bar{u}^{k+1} &= 2u^{k+1} - u^k. \end{aligned}$$

which in this case amounts to

An exercise! :-)

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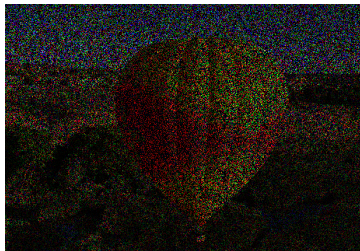
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TV-Inpainting

$$\min P(u) = \min_u \iota_{f_I}(u) + \alpha \|Ku\|_{2,1}$$

with K being a discretization of the color gradient operator, and

$$\iota_{f_I}(u) = \begin{cases} 0 & \text{if } u_i = f_i \text{ for all } i \in I, \\ \infty & \text{otherwise.} \end{cases}$$





We write

$$\min_u P(u) = \min_u \max_p \ell_{f_i}(u) + \langle Ku, p \rangle - \ell_{\|\cdot\|_{2,\infty} \leq \alpha}(p).$$

The (PDHG) updates are

$$\begin{aligned} p^{k+1} &= \text{prox}_{\sigma F^*}(p^k + \sigma K \bar{u}^k) \\ u^{k+1} &= \text{prox}_{\tau G}(u^k - \tau K^* p^{k+1}), \\ \Rightarrow u_i^{k+1} &= \begin{cases} f_i & \text{if } i \in I, \\ (u^k - \tau K^* p^{k+1})_i & \text{otherwise.} \end{cases} \\ \bar{u}^{k+1} &= 2u^{k+1} - u^k. \end{aligned}$$

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$$\min_u P(u) = \min_u \frac{1}{2} \|Au - f\|^2 + \alpha \|Ku\|_{2,1}$$

with K being a discretization of the multichannel gradient operator, A being a convolution with a blur kernel.



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TV-Deblurring - Option 1

We write

$$\min_u P(u) = \min_u \max_p \frac{1}{2} \|Au - f\|^2 + \langle Ku, p \rangle - \iota_{\|\cdot\|_{2,\infty} \leq \alpha}(p).$$

The (PDHG) updates are

$$\begin{aligned} p^{k+1} &= \text{prox}_{\sigma F^*}(p^k + \sigma K \bar{u}^k) \\ u^{k+1} &= \text{prox}_{\tau G}(u^k - \tau K^* p^{k+1}), \\ \bar{u}^{k+1} &= 2u^{k+1} - u^k. \end{aligned}$$

which in this case amounts to

$$\begin{aligned} p^{k+1} &= \underset{p}{\text{argmin}} \frac{1}{2} \|p - (p^k + \sigma K \bar{u}^k)\|^2 + \sigma \iota_{\|\cdot\|_{2,\infty} \leq \alpha}(p), \\ u^{k+1} &= \underset{u}{\text{argmin}} \frac{1}{2} \|u - (u^k - \tau K^* p^{k+1})\|^2 + \frac{\tau}{2} \|Au - f\|^2 \\ &= (I + \tau A^* A)^{-1} (u^k - \tau K^* p^{k+1} + \tau f) \\ \bar{u}^{k+1} &= 2u^{k+1} - u^k. \end{aligned}$$



We write

$$\begin{aligned} & \min_u P(u) \\ &= \min_u \max_{p,q} \langle Au - f, q \rangle - \frac{1}{2} \|q\|^2 + \langle Ku, p \rangle - \iota_{\|\cdot\|_{2,\infty} \leq \alpha}(p) \\ &= \min_u \max_{p,q} \left\langle \begin{pmatrix} A \\ K \end{pmatrix} u, \begin{pmatrix} q \\ p \end{pmatrix} \right\rangle - \langle f, q \rangle - \frac{1}{2} \|q\|^2 - \iota_{\|\cdot\|_{2,\infty} \leq \alpha}(p) \end{aligned}$$

Now we have

$$F^*(p, q) = \langle f, q \rangle + \frac{1}{2} \|q\|^2 + \iota_{\|\cdot\|_{2,\infty} \leq \alpha}(p)$$

$$G(u) = 0$$

$$\tilde{K} = \begin{pmatrix} A \\ K \end{pmatrix}$$



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The (PDHG) updates are

$$q^{k+1} = \operatorname{argmin}_q \frac{1}{2} \|q - (q^k + \sigma A \bar{u}^k)\|^2 + \sigma \langle f, q \rangle + \frac{\sigma}{2} \|q\|^2,$$

$$p^{k+1} = \operatorname{argmin}_p \frac{1}{2} \|p - (p^k + \sigma K \bar{u}^k)\|^2 + \sigma \iota_{\|\cdot\|_{2,\infty} \leq \alpha}(p),$$

$$u^{k+1} = u^k - \tau K^* p^{k+1} - \tau A^* q^{k+1}$$

$$\bar{u}^{k+1} = 2u^{k+1} - u^k.$$

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$$\min_u P(u) = \min_u \frac{1}{2} \|Au - f\|^2 + \alpha \|Ku\|_{2,1}$$

with K being a discretization of the multichannel gradient operator, $A = DB$, with B being a convolution with a blur kernel, and D being a downsampling, e.g. a matrix

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots & \dots \\ 0 & 0 & 0 & 0 & 1 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

PDHG implementation: Option 2 from the previous example.

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TV-Zooming



Input data



Nearest neighbor



TV Zooming



Image Segmentation

$$\min P(u) = \min_u \iota_{\Delta}(u) + \iota_{\geq 0}(u) + \langle u, f \rangle + \alpha \|Ku\|_{2,1}$$

where $K : \mathbb{R}^{n \times m \times c} \rightarrow \mathbb{R}^{nmc \times 2}$ being a discretization of the multichannel gradient operator, and

$$\iota_{\Delta}(u) = \begin{cases} 0 & \text{if } \sum_k u_{i,j,k} = 1, \forall(i,j) \\ \infty & \text{else.} \end{cases}$$
$$\iota_{\geq 0}(u) = \begin{cases} 0 & \text{if } u_{i,j,k} \geq 0, \forall(i,j,k) \\ \infty & \text{else.} \end{cases}$$

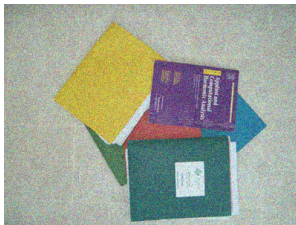
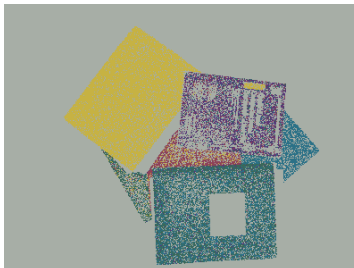
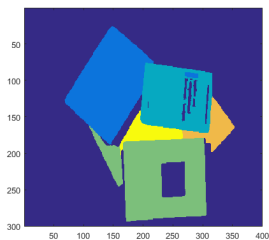
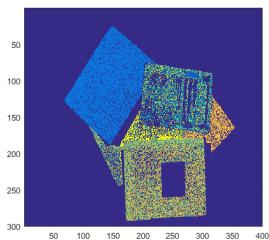


Image Segmentation



Upper row: data term minimization (=kmeans assignment),
lower row: variational method



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Option 1: We solve

$$\min_u \max_p \iota_{\Delta}(u) + \iota_{\geq 0}(u) + \langle u, f \rangle + \langle Ku, p \rangle - \iota_{\|\cdot\|_{2,\infty} \leq \alpha}(p).$$

→ Primal proximal operator: Projection onto unit simplex.

Option 2: We solve

$$\min_u \max_{p,q} \langle Su - 1, q \rangle + \iota_{\geq 0}(u) + \langle u, f \rangle + \langle Ku, p \rangle - \iota_{\|\cdot\|_{2,\infty} \leq \alpha}(p).$$

where $(Su)_{i,j} = \sum_k u_{i,j,k}$.

→ Very simple proximal operators, but additional variable.

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Excursus: Further regularizations and nonconvex minimization through sequences of convex problems.

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Beyond total variation regularization



We used $J(u) = \|Ku\|_{2,1}$ in most applications.

Pros:

- Preserves discontinuities/jumps
- Easy to use and fast to implement
- Good quality on images that are well approximated by piecewise constant images

Cons:

- Staircasing
- Not necessarily a realistic image prior

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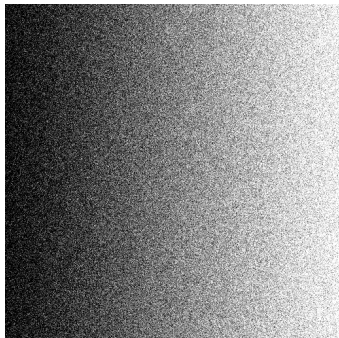
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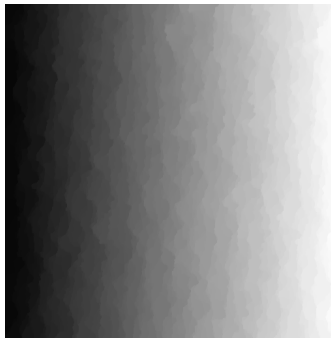
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Total generalized variation³



Input image



TV denoising

Can one avoid this "staircasing" effect while preserving the ability to reconstruct jumps?

³ *Total Generalized Variation*, Bredis, Kunisch, Pock, SIAM Imag. Sci., 2010





Regularization

$$\begin{aligned} TGV(u) &= \min_{Ku=z_1+z_2} \|z_1\|_{2,1} + \beta \|\tilde{K}z_2\|_{2,1} \\ &= \min_z \|Ku - z\|_{2,1} + \beta \|\tilde{K}z\|_{2,1} \end{aligned}$$

with $\tilde{K} : \mathbb{R}^{nm \times 2c} \rightarrow \mathbb{R}^{nm \times 4c}$ being a discretization of the derivative of a vector field.

Idea: Optimally divide into a penalty of the first derivative (TV) and the Hessian.

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Consider the TGV denoising problem

$$\min_{u,z} \frac{1}{2} \|u - f\|_2^2 + \alpha \left(\|Ku - z\|_{2,1} + \beta \|\tilde{K}z\|_{2,1} \right)$$

How can we minimize such a function (with PDHG)?

$$\begin{aligned} & \min_{u,z} \frac{1}{2} \|u - f\|_2^2 + \alpha \left\| \begin{pmatrix} Ku - z \\ \beta \tilde{K}z \end{pmatrix} \right\|_{2,1} \\ &= \min_{u,z} \frac{1}{2} \|u - f\|_2^2 + \alpha \left\| \underbrace{\begin{pmatrix} K & -I \\ 0 & \tilde{K} \end{pmatrix}}_{=:D} \underbrace{\begin{pmatrix} u \\ z \end{pmatrix}}_{=:v} \right\|_{2,1} \\ &= G(v) + F(Dv) \end{aligned}$$

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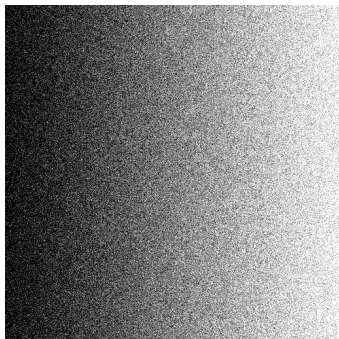
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TGV denoising



Input image



TGV denoising



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Does TGV denoising solve everything?



Input image



TGV denoising

Textures are neither well reconstructed by TV nor by TGV.



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Problems of TV with textures



Input image



TV denoising

Textures are neither well reconstructed by TV nor by TGV.



Nonlocal TV



Input image



NLTV denoising

Idea: Use self-similarity in the image!



Nonlocal TV⁴ based on NL-Means⁵

Construct a similarity graph within an image:

$$w_{i,j} = \exp \left(- \frac{\| \text{patch around pixel } i - \text{patch around pixel } j \|^2}{2\sigma} \right)$$

Anisotropic NLTV regularization:

$$J(u) = \sum_{i,j} \sqrt{w_{i,j}} |u_i - u_j|$$

Illustration of a similar technique:

<http://www.cs.tut.fi/~foi/GCF-BM3D/>

Problem: Construction and size of $W = (w_{i,j}) \in \mathbb{R}^{\#pixels \times \#pixels}$

⁴ *Nonlocal operators with applications to img. proc.*, Gilboa, Osher, 2007

⁵ *A non-local algorithm for image denoising*, Buades, Coll, Morel, 2005



Dictionary learning

Technique that is also based on self-similarity but shifts the problem of constructing a full similarity matrix W :

Dictionary learning: Any patch can be represented as a sparse linear combination of a few dictionary patches.

$$J(u) = \min_{D, \alpha} \|Au - D\alpha\|^2 + \gamma \|\alpha\|_1,$$

where $u \in \mathbb{R}^{n,m,c}$ is an image, $A : \mathbb{R}^{n \times m \times c} \rightarrow \mathbb{R}^{p_1 \times p_2 \times c \times T}$ extracts T color patches of size $p_1 \times p_2$ from the image, and $D \in \mathbb{R}^{p_1 \times p_2 \times c \times t}$ is a matrix or dictionary of t different patches.

The coefficients $\alpha \in \mathbb{R}^{t \times T}$ represent the patches Au .

Usually $p_1 \cdot p_2 \cdot c < t \ll T$.



Dictionary learning

Problems of the form

$$\min_{u, D, \alpha} (E(u, D, \alpha) := G(u) + J(u, D, \alpha)),$$

for

$$J(u, D, \alpha) = \min_{D, \alpha} \|Au - D\alpha\|^2 + \gamma \|\alpha\|_1,$$

are not convex, but at least *biconvex*.

Common strategy: Alternating minimization!

$$(u^{k+1}, \alpha^{k+1}) = \operatorname{argmin}_{u, \alpha} E(u, D^k, \alpha)$$

$$D^{k+1} = \operatorname{argmin}_D E(u^{k+1}, D, \alpha^{k+1})$$

Dictionary learning: See e.g. *K-SVD: An Algorithm for Designing Overcomplete Dictionaries for Sparse Representation*, Aharon, Elad, Bruckstein, 2006, *Sparse representation for color image restoration*, Mairal, Elad, Sapiro, 2008.



Convex optimization is important even for nonconvex optimization!

Landweber method

$$\min_u G(u) + F(u),$$

G is nonconvex, but differentiable, F is simple: Proximal gradient method!

Iteratively-regularized Gauss-Newton / iterative linearization

$$\min_u \frac{1}{2} \|T(u) - f\|^2 + F(Ku),$$

where T is a nonlinear operator:

$$T(u) \approx T(u^k) + \nabla T(u^k) \cdot (u - u^k)$$

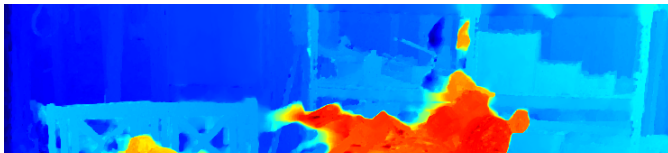
which results in

$$u^{k+1} = \operatorname{argmin}_u \frac{1}{2} \|T(u^k) + \nabla T(u^k) \cdot (u - u^k) - f\|^2 + F(Ku).$$



Stereo Matching

$$\int_{\Omega} |I_1(x + v(x), y) - I_2(x, y)|^2 dx dy + \alpha R(v)$$





Back to PDHG!

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PDHG

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One strongly convex function

We have seen: PDHG works very well on problems of the form

$$\min G(u) + F(Ku),$$

for which F and G are simple.

We get the ergodic convergence of the partial primal-dual gap to zero with rate $\mathcal{O}(1/k)$.

What if our problem is "more friendly"? E.g. what if G or F or both are strongly convex?





Modifications of PDHG

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Case 1: Either G or F^* is strongly convex

Adaptive stepsizes:

$$\begin{aligned} p^{k+1} &= \text{prox}_{\sigma_k F^*}(p^k + \sigma_k K \bar{u}^k), \\ u^{k+1} &= \text{prox}_{\tau_k G}(u^k - \tau_k K^* p^{k+1}), \\ \theta_k &= \frac{1}{\sqrt{1 + 2\gamma\tau_k}}, \\ \tau_{k+1} &= \theta_k \tau_k, \quad \sigma_{k+1} = \sigma_k / \theta_k \\ \bar{u}^{k+1} &= u^{k+1} + \theta_k (u^{k+1} - u^k). \end{aligned} \tag{PDHG2}$$

for $\tau_0 \sigma_0 \leq \|K\|^2$, and G being γ -strongly convex.

Improved convergence because we gain $\frac{\gamma}{2} \|u - u^{n+1}\|^2$ in the subgradient inequality.

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One strongly convex function

Doing an estimate similar to the one of (PDHG) one obtains

$$\|\tilde{u} - u^N\|^2 \leq \tau_N^2 \left(\frac{\|\tilde{u} - u^0\|^2}{\tau_0^2} + \|K\|^2 \|\tilde{p} - p^0\|^2 \right)$$

How fast does τ_N decay?

$$\lim_{N \rightarrow \infty} N\tau_N = \frac{1}{\gamma}$$

Theorem about (PDHG2), strongly convex G , Chambolle, Pock '10

For any $\epsilon > 0$ there exists an N_0 such that for any $N \geq N_0$:

$$\|\tilde{u} - u^N\|^2 \leq \frac{1 + \epsilon}{\gamma^2 N^2} \left(\frac{\|\tilde{u} - u^0\|^2}{\tau_0^2} + \|K\|^2 \|\tilde{p} - p^0\|^2 \right)$$

One strongly convex function

Discussion of the convergence orders:

- Didn't the gradient methods obtain linear convergence on strongly convex energies?
- Yes, but we additionally needed a part of the energy to be L -smooth!
- Note that L -smoothness of the primal corresponds to $1/L$ -strong convexity of the dual!
- So the (PDHG2) is faster (asymptotically $\mathcal{O}(1/N^2)$) than proximal gradient with $\mathcal{O}(1/N)$?
- For the proximal gradient we discussed, yes! But there are adaptive proximal gradient schemes that reach $\mathcal{O}(1/N^2)$!
- Can we massage (PDHG) to have linear convergence if both, G and F^* , are strongly convex?





Two strongly convex functions

Consider

$$\begin{aligned}
 p^{k+1} &= \text{prox}_{\sigma F^*}(p^k + \sigma K \bar{u}^k), \\
 u^{k+1} &= \text{prox}_{\tau G}(u^k - \tau K^* p^{k+1}), \\
 \bar{u}^{k+1} &= u^{k+1} + \theta(u^{k+1} - u^k).
 \end{aligned}
 \tag{PDHG2}$$

Chambolle, Pock '10

For $\mu \leq 2\sqrt{\gamma\delta}/\|K\|$, $\tau = \mu/(2\gamma)$, $\sigma = \mu/(2\delta)$, $\theta \in [1/(1 + \mu), 1]$, G being γ -strongly convex and F^* being δ -strongly convex, there exists $\omega < 1$, such that the iterates of (PDHG2) meet

$$\gamma\|u^N - \tilde{u}\|^2 + (1 - \omega)\delta\|p^N - \tilde{p}\|^2 \leq \omega^N(\gamma\|u^0 - \tilde{u}\|^2 + \delta\|p^0 - \tilde{p}\|^2).$$

→ Linear convergence!

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Understanding the (PDHG) structure

In many image processing and computer vision tasks, we will have to live with the simplest (PDHG) version

$$\begin{aligned}u^{k+1} &= \text{prox}_{\tau G}(u^k - \tau K^* p^k), \\ \bar{u}^{k+1} &= u^{k+1} + (u^{k+1} - u^k), \\ p^{k+1} &= \text{prox}_{\sigma F^*}(p^k + \sigma K \bar{u}^{k+1}),\end{aligned}\tag{PDHG}$$

and convergence of the partial primal-dual gap with $\mathcal{O}(1/k)$.

One thing that still remains somewhat unclear is why the extrapolation in \bar{u}^{k+1} seems to stabilize the algorithm.

→ *Exciting computation on the board!*

